Optimal Retention for a Stop-Loss Reinsurance under the VaR and CTE Risk Measures

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Canadian Institute of Actuaries General Meeting
Chicago, October 19-20, 2006

PD-9: Stochastic Modeling - Current Developments
14:30 - 15:45, October 19, 2006
Outline

- Brief introduction to
  - reinsurance
  - risk measures

- Formulation of optimal reinsurance

- Examples
Reinsurance is a mechanism of transferring risk from an insurer to a second insurance carrier.  

Cedent (or insurer) vs reinsurer

Suppose an insurer is facing an (aggregate) loss $X$

$X$ is a r.v. with survival function $S_X(x) = \Pr(X > x)$

Under Stop-loss reinsurance with retention $d > 0$:

Cedent (insurer):

$$X_I = \begin{cases} 
X, & X \leq d \\
-1, & X > d 
\end{cases} = X \wedge d$$

Reinsurer:

$$X_R = \begin{cases} 
0, & X \leq d \\
X - d, & X > d 
\end{cases} = (X - d)^+,$$
Expected Value Premium Principle:

Net stop-loss premium:

\[ \pi(d) = E[X_R] = E[(X - d)_+] = \int_d^{\infty} S_X(x)dx \]

(Loaded) reinsurance premium:

\[ \delta(d) = (1 + \rho)\pi(d) \]

where \( \rho > 0 \) is the relative safety loading
Total “risk exposure” of the insurer in the presence of the stop-loss reinsurance:

\[ T = X_I + \delta(d) \]

Implications?

Optimal retention \( d \)?

- select \( d \) that optimally minimizes the ruin probability of an insurer,
- optimally maximizes the utility of an insurer.


Our approach is based on risk measures

- Value at Risk (VaR),
- Conditional Tail Expectation (CTE)
Value at Risk vs Conditional Tail Expectation

CTE_X(\alpha) = \mathbb{E}[X | X \geq \text{VaR}_X(\alpha)]

\text{loss distribution } X

\text{probability } \alpha

\text{VaR}_X(\alpha) \equiv S_X^{-1}(\alpha) = F_X^{-1}(1 - \alpha)

\Pr\{X > \text{VaR}_X(\alpha)\} = \alpha \Leftrightarrow \Pr\{X \leq \text{VaR}_X(\alpha)\} = 1 - \alpha
Recall that for a given risk $X$, 

\[ \Pr\{X > \text{VaR}_X(\alpha)\} = \alpha \iff \text{VaR}_X(\alpha) = S_X^{-1}(\alpha) \]

\[ \text{CTE}_X(\alpha) = \mathbb{E}[X|X \geq \text{VaR}_X(\alpha)] \]

\[ T = X_I + \delta(d) \quad \text{where} \quad X_I = X \wedge d \]

Risk measures for the insurer in the presence of stop-loss reinsurance are defined analogously:

\[ \Pr\{X_I > \text{VaR}_{X_I}(d, \alpha)\} = \alpha \]

\[ \Pr\{T > \text{VaR}_T(d, \alpha)\} = \alpha \]

\[ \text{CTE}_{X_I}(d, \alpha) = \mathbb{E}[X_I|X_I \geq \text{VaR}_{X_I}(d, \alpha)] \]

\[ \text{CTE}_T(d, \alpha) = \mathbb{E}[T|T \geq \text{VaR}_T(d, \alpha)] \]
**Optimal Retention**

- **VaR-optimization:**

\[
d^* \rightarrow \text{VaR}_T(d^*, \alpha) = \min_{d > 0}\{\text{VaR}_T(d, \alpha)\}.
\]

- **CTE-optimization:**

\[
\tilde{d} \rightarrow \text{CTE}_T(\tilde{d}, \alpha) = \min_{d > 0}\{\text{CTE}_T(d, \alpha)\}.
\]
Let $p_X$ be the premium payable by the insured to the insurer.

Let $r_X$ be the minimum capital set aside by the insurer so that the insurer’s probability of insolvency is at most $\alpha$; i.e.

$$\Pr\{T > r_X + p_X\} \leq \alpha.$$ 

From the definition of VaR:

$$r_X = \text{VaR}_T(d, \alpha) - p_X.$$
\[ \text{VaR}_{T}(d, \alpha) = \begin{cases} 
  d + \delta(d), & 0 < d \leq S_{X}^{-1}(\alpha), \\
  S_{X}^{-1}(\alpha) + \delta(d), & d > S_{X}^{-1}(\alpha). 
\end{cases} \]

recall that \( \delta(d) = (1 + \rho)\pi(d) \)

Objective:

\[ d^* \rightarrow \text{VaR}_{T}(d^*, \alpha) = \min_{d > 0} \{ \text{VaR}_{T}(d, \alpha) \}. \]
Proof

\[ \text{VaR}_T(d, \alpha) = S_{-1}^{-1}(\alpha) + \delta(d) \]

where \( \rho^* = \frac{1}{1+\rho} \).
The optimal retention \( d^* > 0 \) that minimizes \( \text{VaR}_T(d, \alpha) \) exists if and only if both

\[
\alpha < \rho^* < S_X(0)
\]

and

\[
S_X^{-1}(\alpha) \geq S_X^{-1}(\rho^*) + \delta(S_X^{-1}(\rho^*))
\]

hold, where \( \rho^* = \frac{1}{1 + \rho} \).

When the optimal retention \( d^* \) exists, then \( d^* \) is given by

\[
d^* = S_X^{-1}(\rho^*)
\]

and the minimum VaR of \( T \) is given by

\[
\text{VaR}_T(d^*, \alpha) = d^* + \delta(d^*).
\]
The optimal retention $d^* > 0$ exists if both

$$\alpha < \rho^* < S_X(0)$$

and

$$S_X^{-1}(\alpha) \geq (1 + \rho)E[X]$$

hold
Example: Exponential Distribution

- Assume $\alpha = 0.1$, $\rho = 0.2$
- $X$ is exponentially distributed with $E[X] = 1,000$.
- $S_X(x) = e^{-0.001x}$, $x \geq 0$; $S_X^{-1}(x) = -1,000 \log x$, $0 < x < 1$; and $S_X(0) = 1$.
- $\rho^* = 0.83 > \alpha = 0.1$ \(\checkmark\)
- $S_X^{-1}(\alpha) = -1,000 \log \alpha = 2302.59 > (1 + \rho)E[X] = 1,200$ \(\checkmark\)
- From the Corollary, the optimal retention $d^*$ exists and equals to

$$d^* = S_X^{-1}(\rho^*) = 1,000 \log(1 + \rho) = 182.32.$$
Example: Pareto Distribution

- Similar to the last example, $\alpha = 0.1$ and $\rho = 0.2$

- $X \sim$ Pareto distribution: $S_X(x) = \left( \frac{2,000}{x + 2,000} \right)^3$, $x \geq 0$.

Then $S_X^{-1}(x) = 2,000x^{-1/3} - 2,000$, $0 < x < 1$

- $\rho^* = 0.83 > \alpha = 0.1 \sqrt{\text{}}$

- $S_X^{-1}(\alpha) = 2,000\alpha^{-1/3} - 2,000 = 2308.87$

\[ > (1 + \rho)\mathbb{E}[X] = 1,200 \sqrt{\text{}} \]

- From the Corollary, the optimal retention $d^*$ exists and equals to

\[ d^* = S_X^{-1}(\rho^*) = 125.32 \]
CTE-optimization

\[
\text{CTE}_T(d, \alpha) = \begin{cases} 
  d + \delta(d), & 0 < d \leq S_X^{-1}(\alpha), \\
  S_X^{-1}(\alpha) + \delta(d) + \frac{1}{\alpha} \int_{S_X^{-1}(\alpha)}^{d} S_X(x) \, dx, & d > S_X^{-1}(\alpha).
\end{cases}
\]

The optimal retention \( \tilde{d} > 0 \) that minimizes \( \text{CTE}_T(d, \alpha) \) exists if and only if

\[ 0 < \alpha \leq \rho^* < S_X(0). \]

When the optimal retention \( \tilde{d} > 0 \) exists, \( \tilde{d} \) is given by

\[ \tilde{d} = S_X^{-1}(\rho^*) \quad \text{if} \quad \alpha < \rho^*, \]

and

\[ \tilde{d} \geq S_X^{-1}(\rho^*) \quad \text{if} \quad \alpha = \rho^*, \]
Assume $\alpha = 0.1$, $\rho = 2.7$ and $X$ has the same exponential distribution as in the earlier example.

Then,

$$S_X^{-1}(\alpha) - \left( S_X^{-1}(\rho^*) + (1 + \rho) \int_{S_X^{-1}(\rho^*)}^{\infty} S_X(x) \, dx \right) = -5.74773 < 0.$$ 

the optimal retention $d^*$ does not exist (VaR criterion).

But the optimal retention $\tilde{d}$ exists under CTE criterion since

$$\rho^* = 0.27 > \alpha = 0.1$$

$$\tilde{d} = S_X^{-1}(\rho^*) = 1308.33.$$
Example: Pareto Distribution

- Similarly we set $\alpha = 0.1$, $\rho = 2.7$ except that $X$ has the same Pareto distribution.

- Then,

$$S_X^{-1}(\alpha) - \left( S_X^{-1}(\rho^*) + (1 + \rho) \int_{S_X^{-1}(\rho^*)}^{\infty} S_X(x) \, dx \right) = -331.172 < 0.$$  

- Again, the optimal retention $d^*$ does not exist.

- However, the optimal retention $\tilde{d}$ exists since

$$\rho^* = 0.27 > \alpha = 0.1$$

$$\tilde{d} = S_X^{-1}(\rho^*) = 1093.36$$
The optimal retention has a very simple analytic form

If optimal solutions exist, then both VaR- and CTE-based optimization criteria yield the same optimal retentions, except when $\alpha = \rho^*$

$$d^* = \tilde{d} = S_X^{-1} \left( \frac{1}{1 + \rho} \right)$$

The optimal retention depends only on the assumed loss distribution and the reinsurer’s safety loading factor

The CTE criterion is preferred to the VaR criterion

Limitations?
Weng, Yi, Cai and Tan (2006) generalize these results to

**VaR-optimization**:

\[
\text{VaR}_{T_f^*}(X)(\alpha) = \min_{f \in \mathcal{F}} \left\{ \text{VaR}_{T_f}(X)(\alpha) \right\}
\]

**CTE-optimization**:

\[
\text{CTE}_{T_f^*}(X)(\alpha) = \min_{f \in \mathcal{F}} \left\{ \text{CTE}_{T_f}(X)(\alpha) \right\}
\]

where \( \mathcal{F} \) denotes the class of ceded loss functions, which consists of all increasing convex functions \( f(x) \) defined on \([0, \infty)\) and satisfying \( 0 \leq f(x) \leq x \) for \( x \geq 0 \).

Our results show that depending on \( \alpha \) and \( \rho \), a stop-loss reinsurance is optimal in some cases while a quota-share reinsurance or the combination of a stop-loss reinsurance and a quota-share reinsurance is optimal in some other cases.
References


