SESSION/SÉANCE :

37 – Applications of Forward Mortality Factor Models in Life Insurance Practice

SPEAKER(S)/CONFÉRENCIER(S) :

Nan Zhu, Georgia State University and Illinois State University
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1. INTRODUCTION

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MOTIVATION

- Consider a 60 year old individual who faces *systematic* mortality risk:
  1. Currently expected to die with probability of 0.7% next year (mortality rate). How uncertain is this appraisal? What are the chances that the rate is 0.65% or 0.75%?
    - Risk in mortality rates
  2. Currently expected to live 25.1 more years. How uncertain is this number? What are the chances that it changes to 23.7 or 24.5 next year?
    - Risk in mortality projections

- While the two questions are related, and the distinction does not matter in theory, it is relevant for the *econometrical/statistical* approach
- For personal financial planning/household finance, insurers’ liability risk evaluation, and government/public economics, question 2 may be more suitable
MOTIVATION

- Current stochastic mortality models focus on stochastically forecasting mortality rates (question 1)
  - red in graphic
- This essay considers the risk in mortality projections (question 2)
  - blue in graphic
MOTIVATION

Challenges for the application of stochastic mortality models in life insurance practice:

- Incompatibility with "classical life contingencies theory", which presents backbone of insurers’ EDP systems
- Complexity of many of the existing approaches

- Increasing discrepancy between life insurance research and actuarial practice
- Potential reason for sluggish development of the longevity-linked capital market: Stochastic methods necessary to assess company’s capital relief when hedging part of their exposure

- **Forward mortality factor models** as a possible solution
LITERATURE REVIEW: MORTALITY

Large literature with various methods on mortality forecasting:

Lee-Carter approach (Lee and Carter (JASA, 1992)):

$$\log m(t, x) = \beta_x^{(1)} + \beta_x^{(2)} \kappa_t^{(2)}$$

CBD-Perks model (Cairns et al. (JRI, 2006)):

$$\logit q(t, x) = \kappa_t^{(1)} + \kappa_2^{(2)} (x - \bar{x})$$

P-splines method (Currie et al. (Statistical Modeling, 2004)):

non-parametric model

All these methodologies...

... rely on past mortality data to project the *mortality experience* in some optimal sense

... pay little attention on the uncertainty (error estimates) associated with the *projections*

... may fail to identify the transiency of different random sources
IDEA

Look at mortality forecasts generated from different windows of past mortality experience using a fixed projection methodology

⇒ our "raw" data: time series of mortality forecasts

\[ \{ \tau p_x(t) | (\tau, x) \in C \} \]_{t=0,1,2,...}

- Functions in two variables: age \( x \) and time horizon \( \tau \)
- Dynamic stochastic models described by (infinite-dimensional) stochastic (difference) equation
- Know semi-parametric representation of all models that can be realized by Normal-Distributed random vectors
- Apply principle component analysis (PCA) for model identification & Maximum likelihood approach for estimation

Applications in life insurance context
- Economic capital for a stylized life insurance company
- Guaranteed Annuity Options
- Guaranteed Minimum Income Benefits within Variable Annuities
1. Introduction

2. FORWARD MORTALITY FACTOR MODELS

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4. Application II: Valuation of Annuitization Options

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FORWARD FORCE OF MORTALITY FRAMEWORK

Forward survival probabilities:

\[
\tau \rho_x(t, T + \tau) \mathbf{1}_{\{\tau_{x-T} > T\}} = \mathbb{E}^P \left[ \mathbf{1}_{\{\tau_{x-T} > T+\tau\}} | \mathcal{F}_t \lor \{\tau_{x-T} > T\} \right], \quad 0 \leq T \leq t \leq T + \tau
\]

Forward force of mortality (easier to model/work with than \(\tau \rho_x(t, T + \tau)\)):

\[
\mu_t(\tau, x) = -\frac{\partial}{\partial \tau} \log \{\tau \rho_x(t, t + \tau)\}
\]

Consider time-homogenous diffusion-driven models (cf. Bauer et al. (2010))

\[
d\mu_t = (A \mu_t + \alpha_t) \, dt + \sigma_t \, dW_t
\]

Drift condition (Cairns et al. (2006, ASTIN)): With \(W_t\) Brownian motion under \(\mathbb{P}\),

\[
\alpha_t(\tau, x) = \sigma_t(\tau, x) \times \int_0^\tau \sigma'_t(s, x) \, ds
\]

Bauer et al. (2010): \(\mu_t\) allows for a Gaussian finite-dimensional realization (FDR) if

\[
\sigma(\tau, x) = C(x + \tau) \times \exp\{M\tau\} \times N
\]
DATA AND CALIBRATION

Generation of underlying mortality forecasts
- Lee-Carter model \(\rightarrow\) 31 generation life tables (1977-2007)

Specification & Calibration
- Principle component analysis
  - The first principle component explains 91% of total variation \(\rightarrow\) one-factor model
  - Shape of principle component can be adequately described by Logistic-Gompertz specification:
    \[
    \sigma(\tau, x) = k \frac{\exp\{c(x + \tau) + |d|\}}{1 + \exp\{c(x + \tau) + d\}} (a + \tau) e^{-b\tau}
    \]

Maximum likelihood estimation \(\Rightarrow\) parameter estimates \((k, c, d, a, b)\)
• Confidence intervals for life expectancies in one year for a now 20 year old female (USA)

• Comparison with conventional mortality forecasting approach – Lee-Carter
  ➢ Generally underestimate the risk in mortality projections
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4. Application II: Valuation of Annuitization Options
5. Conclusion
MODEL FRAMEWORK

- Available Capital at time zero: $AC_0 = E$
- Available Capital at time one: $AC_1 = E_Q[\text{Assets}|\mathcal{F}_1] - E_Q[\text{Liabilities}|\mathcal{F}_1]$
- One-year mark-to-market approach for calculating Economic Capital:

\[
EC = \rho \left( AC_0 - AC_1 \right) \rho(0, 1)
\]

\(\rho\): monetary risk measure \((L^2(\Omega, \mathcal{F}, \mathbb{P}) \rightarrow \mathbb{R})\)

- Solvency Capital Requirement (Solvency II):

\[
EC = \text{SCR} = \text{VaR}_\alpha(L) = \arg\min_x \{ \mathbb{P}(L > x) \leq 1 - \alpha \}
\]

- Conditional Tail Expectation (used within SST):

\[
EC = \text{CTE}_\alpha = \mathbb{E}[L|L \geq \text{VaR}_\alpha(L)]
\]

- Compare two cases: without and with systematic mortality risk
A STYLIZED LIFE INSURANCE COMPANY

• Newly founded life insurer using Equivalence Principle for pricing (no profits, expenses), risk-neutral w.r.t. mortality risk
• Portfolio of policies: Term-life, endowment, life annuities
• Financial Portfolio: Invest in stock, 1-year, 3-year, 5-year, and 10-year govt. bonds with equal weight
  ➢ Financial market model: Extended Black-Scholes model with stochastic interest rates (Vasicek model)
  ➢ Calibrated to UK data from 06-1988 to 06-2008 using a Kalman filter

• 2,500,000 simulations of $A_1$ and $V_1 \Rightarrow AC_1 \Rightarrow EC$
• Advantages of Forward Mortality Factor Models:
  ➢ No nested simulations necessary
  ➢ Oeod to simulate $(Z_s)_{0\leq s\leq 1}$ instead of the entire mortality surface
PORTFOLIO OF THE COMPANY
(E = 2,000,000)

<table>
<thead>
<tr>
<th>x</th>
<th>i</th>
<th>( n^\text{term/end,ann}_{x,i} )</th>
<th>( B^\text{term/end,ann}_{x,i} )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Term Life</td>
<td></td>
</tr>
<tr>
<td>30</td>
<td>20</td>
<td>250</td>
<td>100,000</td>
</tr>
<tr>
<td>35</td>
<td>15</td>
<td>250</td>
<td>100,000</td>
</tr>
<tr>
<td>40</td>
<td>10</td>
<td>250</td>
<td>100,000</td>
</tr>
<tr>
<td>45</td>
<td>5</td>
<td>250</td>
<td>100,000</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Endowment</td>
<td></td>
</tr>
<tr>
<td>40</td>
<td>20</td>
<td>500</td>
<td>50,000</td>
</tr>
<tr>
<td>45</td>
<td>15</td>
<td>500</td>
<td>50,000</td>
</tr>
<tr>
<td>50</td>
<td>10</td>
<td>500</td>
<td>50,000</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Annuities</td>
<td></td>
</tr>
<tr>
<td>60</td>
<td>(35)</td>
<td>250</td>
<td>18,000</td>
</tr>
<tr>
<td>70</td>
<td>(25)</td>
<td>250</td>
<td>18,000</td>
</tr>
</tbody>
</table>
EMPIRICAL CDF OF ONE-YEAR LOSS
ECONOMIC CAPITAL FOR THE STYLIZED COMPANY

<table>
<thead>
<tr>
<th>Confidence level $\alpha$</th>
<th>VaR</th>
<th>standard error</th>
<th>CTE</th>
<th>standard error</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Deterministic mortality</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>90%</td>
<td>3,995,515</td>
<td>39,596</td>
<td>5,712,494</td>
<td>41,787</td>
</tr>
<tr>
<td>99%</td>
<td>7,804,053</td>
<td>93,629</td>
<td>9,047,340</td>
<td>106,028</td>
</tr>
<tr>
<td><strong>Stochastic mortality</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>90%</td>
<td>4,494,682</td>
<td>41,753</td>
<td>6,457,341</td>
<td>45,018</td>
</tr>
<tr>
<td>99%</td>
<td>8,844,264</td>
<td>98,542</td>
<td>10,306,897</td>
<td>118,670</td>
</tr>
</tbody>
</table>

- For the 90% (99%) threshold, the Value-at-Risk increases by around 12% (13%)
- For the 90% (99%) threshold, the Conditional Tail Expectation increases by around 13% (14%)
- **Economic importance of mortality risk**
CALCULATIONS OF VaR<sub>99%</sub>: JOINTLY & SEPARATELY

<table>
<thead>
<tr>
<th>Evaluation technique</th>
<th>stand alone</th>
<th>capital allocation</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Deterministic mortality</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Term Life</td>
<td>338,711</td>
<td>27,343</td>
</tr>
<tr>
<td>Endowment</td>
<td>2,624,822</td>
<td>1,697,119</td>
</tr>
<tr>
<td>Annuities</td>
<td>5,961,906</td>
<td>6,079,590</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td>8,925,439</td>
<td>7,804,053</td>
</tr>
<tr>
<td><strong>Stochastic mortality</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Term Life</td>
<td>473,474</td>
<td>-120,123</td>
</tr>
<tr>
<td>Endowment</td>
<td>2,652,375</td>
<td>1,458,029</td>
</tr>
<tr>
<td>Annuities</td>
<td>7,670,600</td>
<td>7,506,357</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td>10,796,450</td>
<td>8,844,264</td>
</tr>
</tbody>
</table>

- Under systematic risk, **negative** capital allocated to term business
  ⇒ Natural hedging opportunity?
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5. Conclusion
GUARANTEED ANNUITY OPTIONS (GAO)

- Option within traditional endowment contract to choose – at maturity – between fixed payout (normalized to one) and annuity guaranteed at time zero

- Value at maturity if alive: \( V_T^{GAO} = \max \left\{ 1, g_{x_0, T}^{GAO} \bar{a}_{x_0 + T}(T) \right\} = 1 + C_T^{GAO} \)

- Value at time zero: \( V_0^{GAO} = p(0, T) T p_{x_0}(0, T) + C_0^{GAO} \)

- By change of numéraire technique and "freezing" weights akin to LIBOR market models (Brace et al. (1997, Mathematical Finance)), we derive

\[
C_0^{GAO} \approx X_0 \left( g_{x_0, T}^{GAO} \Phi \left( -d_2^{GAO} \right) - \frac{p(0, T) T p_{x_0}(0, T)}{X_0} \Phi \left( -d_1^{GAO} \right) \right),
\]

where

\[
d_1^{GAO} = \frac{\log \left\{ \frac{p(0,T) T p_{x_0}(0,T)}{X_0 g_{x_0, T}^{GAO}} \right\} + \frac{1}{2} \sigma_{GAO}^2}{\sigma_{GAO}},
\]

\[
d_2^{GAO} = d_1^{GAO} - \sigma_{GAO},
\]

\[
\sigma_{GAO}^2 = \int_0^T \| \gamma(u, T, x_0) \|^2 du
\]
RESULTS FOR GAO

<table>
<thead>
<tr>
<th>$(x_0, T)$</th>
<th>$g_{x_0,T}^{GAO}$</th>
<th>$V_0^{GAO}$</th>
<th>$C_0^{GAO}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(30, 35)$</td>
<td>0.0803</td>
<td>0.1315</td>
<td>0.0050</td>
</tr>
<tr>
<td>$(40, 25)$</td>
<td>0.0825</td>
<td>0.2366</td>
<td>0.0088</td>
</tr>
<tr>
<td>$(50, 15)$</td>
<td>0.0850</td>
<td>0.4242</td>
<td>0.0152</td>
</tr>
</tbody>
</table>

**Deterministic mortality**

**Stochastic mortality**

- Significant impact of stochastic mortality
GUARANTEED MINIMUM INCOME BENEFITS (GMIBs) WITHIN VA

- Option within Variable Annuity to choose – at maturity – between account value AT or – at time zero – guaranteed annuity payment $g_{x_0,T}^{GMIB}$

\[
V_{0}^{GMIB} = E^Q \left[ \mathbf{1}_{\{T_{X_0} > T\}} e^{-\int_{0}^{T} r_s ds} \max \left\{ g_{X_0,T}^{GMIB} A_0 \bar{a}_{X_0+T}(T), A_T \right\} \right] \\
+ \sum_{k=0}^{T-1} E^Q \left[ \mathbf{1}_{\{T_{X_0} \leq [k,k+1)\}} e^{-\int_{0}^{k+1} r_s ds} A_{k+1} \right] \\
= E^Q \left[ e^{-\int_{0}^{T} r_s + \mu_s(0,X_0+s) ds} \max \left\{ g_{X_0,T}^{GMIB} A_0 \sum_{k=T}^{\infty} p(T, k - T) k - T p_{X_0+T}(T,k), A_T \right\} \right] \\
+ A_0 e^{-\phi T} \left( 1 - T p_{X_0}(0, T) \right)
\]

- Advantage of Forward Mortality Factor Models (Again!): No nested simulations necessary
RESULTS FOR GMIB

Calculate the *fair continuously deducted option fee*

<table>
<thead>
<tr>
<th></th>
<th>Deterministic mortality</th>
<th>Stochastic mortality</th>
</tr>
</thead>
<tbody>
<tr>
<td>((x_0, T) = (30, 35))</td>
<td>0.0278</td>
<td>0.0319</td>
</tr>
<tr>
<td>((x_0, T) = (40, 25))</td>
<td>0.0321</td>
<td>0.0349</td>
</tr>
<tr>
<td>((x_0, T) = (50, 15))</td>
<td>0.0423</td>
<td>0.0445</td>
</tr>
</tbody>
</table>

- Modest impact compared with GAO
  ⇒ Equity risk dominates mortality risk

- Mortality risk affects different mortality-contingent options dissimilarly
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CONCLUSION

Applications of forward mortality factor models in life insurance practice:
• Economic capital of a life insurer  
  ⇒ Considerable effect on solvency
• Guaranteed Annuity Options  
  ⇒ Values increase greatly with systematic mortality risk
• Guaranteed Minimum Income Benefits  
  ⇒ Modest increase in the fair option fees

Advantages of model family:
• Compatibility with classical actuarial theory
• Tractability of forward mortality factor models
• Numerical advantages – in particular, no nested simulations
CONTACT

Nan Zhu
nzhul@student.gsu.edu
Georgia State University

https://sites.google.com/site/nanzhugsu/
& www.rmi.gsu.edu

Thank you!